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Higher order terms in the non-abelian D-brane effective action and magnetic background fields

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Abstract: Recently a proposal for the bosonic sector of the non-abelian effective D-brane action was given through order α'^4. As the resulting expressions turned out to be quite involved, some checks of this result are called for. In the present paper we calculate the spectrum in the presence of constant magnetic background fields and compare it to the string theoretical result. Apart from a small typo in the original expression (the overall sign of the α'^4 term), we obtain perfect agreement. We discuss potential applications.

Keywords: D-branes, M(atrix) Theories.

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1. Introduction

The effective action for D-branes is one of the few tools available for the study of the dynamics of D-branes. It is quite surprising that, in the limit of slowly varying fields, the effective action for a single Dp-brane is known to all orders in α'. It is given by the ten dimensional supersymmetric Born-Infeld action dimensionally reduced to $p+1$ dimensions [1, 2].

No such a result is presently available for the case of several, say n, coinciding Dp-branes. In leading order in α', it is the ten-dimensional $N = 1$ supersymmetric $U(n)$ Yang-Mills theory dimensionally reduced to $p+1$ dimensions [3]. There are no $\mathcal{O}(\alpha')$ corrections. The bosonic $\mathcal{O}(\alpha'^2)$ corrections were first obtained in [4] and [5]. The fermionic terms through this order were obtained in [6] and [7]. In [6], supersymmetry fixed the correction while in [7] a direct calculation starting from four-point open superstring amplitudes was used. Requiring the existence of certain BPS configurations, called stable holomorphic bundles [8], allows for a selfconsistent determination of the effective action [9]. This was applied in [10] to determine the bosonic $\mathcal{O}(\alpha'^3)$ terms in the effective action. In [11], supersymmetry was used not only to confirm the results of [10] but to construct the terms quadratic in the gauginos through this order as well. Later on, these results were confirmed through a direct calculation of five point functions in open superstring theory [12]. Restricting to the special case of four dimensions, one finds that, through this order, the effective action also coincides with the one loop effective action in $N = 4$, $d = 4$ super Yang-Mils [13]–[15].

Recently, the methods of [9] were used to determine the bosonic terms in the effective action through order α'^4 [16]. Through this order, the effective action is given by,

$$\mathcal{L} = \frac{1}{g^2} \left(\mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 \right), \tag{1.1}$$

where the leading term is simply[1]

$$\mathcal{L}_0 == -\mathrm{Tr}\left\{\frac{1}{4}F^2\right\}. \tag{1.2}$$

Subsequently we have

$$\mathcal{L}_2 = \mathrm{STr}\left\{\frac{1}{8}F^4 - \frac{1}{32}F^2F^2\right\}, \tag{1.3}$$

where STr denotes the symmetrized trace prescription. At this point both the overall multiplicative factor in front of the action as well as the scale of the gauge fields got fixed [10]. The next term is[2]

$$\mathcal{L}_3 = \frac{\zeta(3)}{2\pi^3}\mathrm{Tr}\left\{[D_3,D_2]D_4F_{51}\,D_5[D_4,D_3]F_{12}\right\}. \tag{1.4}$$

The overall coefficient of this term remained undetermined when using the method of [9]. It was fixed by comparing it to the partial result for this term in [17] which was obtained by a direct string theoretic calculation. Note that this expression is considerably simpler than the one which originally appeared in [10]. This is due to a different choice of basis in which we express the action. Indeed, using partial integration, Bianchi identities, ..., the action can be written in numerous different ways. Finally the fourth order term is completely determined by the method of [9] and it is given by [16],

$$\mathcal{L}_4 = \mathcal{L}_{4,0} + \mathcal{L}_{4,2} + \mathcal{L}_{4,4}\,, \tag{1.5}$$

with

$$\begin{aligned}
\mathcal{L}_{4,0} &= -\mathrm{STr}\Bigg(\frac{1}{12}F_{12}F_{23}F_{34}F_{45}F_{56}F_{61} - \frac{1}{32}F_{12}F_{23}F_{34}F_{41}F_{56}F_{65} + \\
&\qquad + \frac{1}{384}F_{12}F_{21}F_{34}F_{43}F_{56}F_{65}\Bigg), \\
\mathcal{L}_{4,2} &= -\frac{1}{48}\mathrm{STr}\Big(-2\,F_{12}D_1D_6D_5F_{23}D_6F_{34}F_{45} - \\
&\qquad - F_{12}D_5D_6F_{23}D_6D_1F_{34}F_{45} + \\
&\qquad + 2\,F_{12}\,[D_6,D_1]\,D_5F_{23}F_{34}D_4F_{56} + \\
&\qquad + 3\,D_4D_5F_{12}F_{23}\,[D_6,D_1]\,F_{34}F_{56} + \\
&\qquad + 2\,D_6\,[D_4,D_5]\,F_{12}F_{23}D_1F_{34}F_{56} \\
&\qquad + 2\,D_6D_5F_{12}\,[D_6,D_1]\,F_{23}F_{34}F_{45} + \\
&\qquad + 2\,[D_6,D_1]\,D_3D_4F_{12}F_{23}F_{45}F_{56} + \\
&\qquad + [D_6,D_4]\,F_{12}F_{23}\,[D_3,D_1]\,F_{45}F_{56}\Big),
\end{aligned}$$

[1]Most of the time, we put $2\pi\alpha' = 1$. Our metric follows the "mostly plus" convention. The $u(n)$ generators are always anti-hermitean and we use the following notation: $F^m \equiv F_{a_1}{}^{a_2}F_{a_2}{}^{a_3}\cdots F_{a_m}{}^{a_1} \equiv F_{a_1a_2}F_{a_2a_3}\cdots F_{a_ma_1} \equiv F_{12}F_{23}\cdots F_{m1}$.

[2]All results are of course modulo field redefinition terms.

$$\begin{aligned}\mathcal{L}_{4,4} = -\frac{1}{1440}\mathrm{STr}\Big(& D_6[D_4,D_2]D_5D_5[D_1,D_3]D_6F_{12}F_{34} + \\ & + 4\,D_2D_6[D_4,D_1][D_5,[D_6,D_3]]D_5F_{12}F_{34} + \\ & + 2\,D_2[D_6,D_4][D_6,D_1]D_5[D_5,D_3]F_{12}F_{34} + \\ & + 6\,D_2[D_6,D_4]D_5[D_6,D_1][D_5,D_3]F_{12}F_{34} + \\ & + 4\,D_6D_5[D_6,D_4][D_5,D_1][D_4,D_3]F_{12}F_{23} + \\ & + 4\,D_6D_5[D_4,D_2][D_6,D_1][D_5,D_3]F_{12}F_{34} + \\ & + 4\,D_6[D_5,D_4][D_3,D_2][D_5,[D_6,D_1]]F_{12}F_{34} + \\ & + 2\,[D_6,D_1][D_2,D_6][D_5,D_4][D_5,D_3]F_{12}F_{34}\Big)\,. \end{aligned} \tag{1.6}$$

The overall sign of $\mathcal{L}_4$ is different from the one in [16]. This is due to a typo in [16]. Obviously, the expression for the $\mathcal{O}(\alpha'^4)$ terms is very involved. So an independent check of these is called for. In [18], further developed in [19] and [20], such a test was proposed. One starts from two D$2p$-branes wrapped around a $2p$-dimensional torus. When switching on constant magnetic background fields, this yields, upon T-dualizing, two intersecting Dp-branes. String theory allows for the calculation of the spectrum of strings stretching between different branes [21, 22]. In the context of the effective action, the spectrum should be reproduced by the mass spectrum of the off-diagonal gauge field fluctuations. In [23] it was shown that the bosonic terms through $\mathcal{O}(\alpha'^3)$ correctly reproduce the spectrum of the gauge fields. In [24], the method was further extended such that the fermionic terms could be tested as well. In the present paper we turn to the test of the bosonic terms at order α'^4. This is particularly interesting, as it is precisely at this order that the mass spectrum such as obtained from the symmetrized trace prescription for the non-abelian Born-Infeld [25] (this corresponds to $\mathcal{L}_{4,0}$ in eq. (1.6)) starts to deviate from the string theoretic spectrum [18]–[20].

2. The spectrum from string theory

We consider a constant magnetic background on two coincident D$2p$-branes,

$$\mathcal{F}_{2a-1\,2a} = i\begin{pmatrix} F_a & 0 \\ 0 & -F_a \end{pmatrix}, \tag{2.1}$$

with $a \in \{1,2,\dots,p\}$ and $F_a \in \mathbb{R}$, $F_a > 0$. We choose a gauge such that $\mathcal{A}_{2a-1} = 0$, $\forall a$, and T-dualize in the 2, 4, …, $2p$ directions. We end up with two intersecting Dp-branes. Taking the first brane located along the 1, 3, …, $2p-1$ directions, one finds that the other brane has been rotated with respect to the first one over an angle θ_1 in the 12 plane, over an angle θ_2 in the 34 plane, …, over an angle θ_p in the $2p-1\,2p$ plane. The angles are determined by the magnetic fields,

$$\theta_a = 2\,\arctan 2\pi\alpha' F_a, \quad \forall a \in \{1,2,\dots,p\}. \tag{2.2}$$

One finds for the mass of the open strings stretching between the two branes [22, 18, 19],

$$M^2 = \frac{1}{2\pi\alpha'}\left(\sum_{b=1}^{p}(2\,m_b+1)\theta_b \pm 2\,\theta_a\right), \quad a \in \{1,\dots,p\}, \quad m_b \in \mathbb{N}. \tag{2.3}$$

In the previous, we temporarily reinstated the factors of $2\pi\alpha'$.

3. The spectrum from the effective action

The mass formula given in eq. (2.3) should be reproduced by the effective action. Taking the effective action given in eqs. (1.2)–(1.6), one turns on the magnetic background given in in eq. (2.1) and one subsequently diagonalizes the linearized equations of motion for the off-diagonal fluctuations. Expanding eq. (2.3) in powers of α' using eq. (2.2) and setting $2\pi\alpha'$ back to one, we get

$$M^2 = \sum_{b=1}^{p} 2(2\,m_b+1)\left(F_b - \frac{F_b^3}{3} + \frac{F_b^5}{5}\right) \pm 4\left(F_a - \frac{F_a^3}{3} + \frac{F_a^5}{5}\right) + \mathcal{O}(F^7)\,. \tag{3.1}$$

From this it is clear that the terms linear in F have to be reproduced by $\mathcal{L}_0$, those cubic in F by $\mathcal{L}_2$, $\mathcal{L}_3$ should not contribute to the spectrum and $\mathcal{L}_4$ is responsible for the terms quintic in F in the spectrum.

3.1 Leading order result

We turn on a constant magnetic background $\mathcal{F}_{ab}$ with the corresponding background gauge potentials $\mathcal{A}_a$. We parameterize the gaugefields by $A_a = \mathcal{A}_a + \delta A_a$. As the calculation of the spectrum only probes $U(2)$ sub-sectors of the full $U(n)$ theory [27], we take $U(2)$ as the gauge group. We compactify $2p$ dimensions on a torus and introduce complex coordinates for the compact directions, $z^\alpha = (x^{2\alpha-1} - i\,x^{2\alpha})/\sqrt{2}$, $\bar{z}^{\bar{\alpha}} = (z^\alpha)^*$, $\alpha \in \{1,\ldots,p\}$. We take magnetic fields in the compact directions, such that $\mathcal{F}_{\alpha\beta} = \mathcal{F}_{\bar{\alpha}\bar{\beta}} = 0$, $\mathcal{F}_{\alpha\bar{\beta}} = 0$ for $\alpha \neq \beta$ and[3]

$$\mathcal{F}_{\alpha\bar{\alpha}} = i\begin{pmatrix} f_\alpha & 0 \\ 0 & -f_\alpha \end{pmatrix}, \tag{3.2}$$

where the f_α, $\alpha \in \{1,\ldots,p\}$ are imaginary constants such that $if_\alpha = F_\alpha > 0$. We are only interested in the off-diagonal components of the gauge fields,

$$\delta A = i\begin{pmatrix} 0 & \delta A^+ \\ \delta A^- & 0 \end{pmatrix}, \tag{3.3}$$

as the diagonal fluctuations probe the abelian part of the action. The spectrum for δA^+ is equal to that of δA^-, which reflects the two orientations of the strings stretching between the two branes. Throughout the paper, we will investigate the spectrum for δA^+.

Linearizing the equations of motion which follow from $\mathcal{L}_0$ in eq. (1.2), we get,

$$\begin{aligned} 0 &= \left(\mathcal{D}^2 + 4if_\alpha\right)\delta A^+_\alpha - \sum_\beta \mathcal{D}_\alpha(\mathcal{D}_\beta \delta A^+_{\bar{\beta}} + \mathcal{D}_{\bar{\beta}}\delta A^+_\beta)\,, \\ 0 &= \left(\mathcal{D}^2 - 4if_\alpha\right)\delta A^+_{\bar{\alpha}} - \sum_\beta \mathcal{D}_{\bar{\alpha}}(\mathcal{D}_\beta \delta A^+_{\bar{\beta}} + \mathcal{D}_{\bar{\beta}}\delta A^+_\beta)\,, \end{aligned} \tag{3.4}$$

where

$$\mathcal{D}^2 \delta A^+ = \left(\Box_{NC} + \sum_\beta \mathcal{D}_\beta \mathcal{D}_{\bar{\beta}} + \sum_\beta \mathcal{D}_{\bar{\beta}}\mathcal{D}_\beta\right), \tag{3.5}$$

[3] We do not sum over repeated indices corresponding to *complex* coordinates, unless indicated otherwise.

where $\Box_{NC}$ denotes the d'alambertian in the non-compact directions and we have

$$\mathcal{D}_\alpha \delta A^+ = (\partial_\alpha + 2i\mathcal{A}_\alpha)\,\delta A^+ \,, \qquad \mathcal{D}_{\bar\alpha} \delta A^+ = (\partial_{\bar\alpha} + 2i\mathcal{A}_{\bar\alpha})\,\delta A^+ \,. \tag{3.6}$$

Using

$$[\mathcal{D}_\alpha, \mathcal{D}_{\bar\beta}] = 2i\delta_{\alpha\beta} f_\alpha \,, \tag{3.7}$$

and choosing the gauge,

$$\sum_\beta (\mathcal{D}_\beta \delta A^+_{\bar\beta} + \mathcal{D}_{\bar\beta}\delta A^+_\beta) = 0 \,, \tag{3.8}$$

we can rewrite eq. (3.4) as

$$\begin{aligned} 0 &= \left(\Box_{NC} + 2\sum_\beta \left(\mathcal{D}_\beta \mathcal{D}_{\bar\beta} - if_\beta\right) + 4if_\alpha \right) \delta A^+_\alpha \,, \\ 0 &= \left(\Box_{NC} + 2\sum_\beta \left(\mathcal{D}_\beta \mathcal{D}_{\bar\beta} - if_\beta\right) - 4if_\alpha \right) \delta A^+_{\bar\alpha} \,. \end{aligned} \tag{3.9}$$

In order to diagonalize this, we introduce a complete set of functions on the torus,

$$\phi_{\{m_1,m_2,\ldots,m_p\}}(z,\bar z) \equiv \mathcal{D}^{m_1}_{z^1}\mathcal{D}^{m_2}_{z^2}\ldots\mathcal{D}^{m_p}_{z^p}\,\phi_{\{0,0,\ldots,0\}}(z,\bar z)\,, \tag{3.10}$$

where $\phi_{\{0,0,\ldots,0\}}$ is defined through,

$$\mathcal{D}_{\bar\alpha}\,\phi_{\{0,0,\ldots,0\}}(z,\bar z) = 0, \qquad \forall\,\bar\alpha \in \{1,2,\ldots,p\}\,. \tag{3.11}$$

The function $\phi_{\{0,0,\ldots,0\}}(z,\bar z)$ was explicitely constructed in [26] and [27]. It is fully determined by eq. (3.11) and the requirement that they satisfy proper boundary conditions. Denoting the non-compact coordinates collectively by y, we make the expansion,

$$\delta A^+(y,z,\bar z) = \sum_{(m_1,\ldots,m_p)\in\mathbb{N}^p} \delta A^{+\{m_1,\ldots,m_p\}}(y)\,\phi_{\{m_1,\ldots,m_p\}}(z,\bar z)\,. \tag{3.12}$$

Using eq. (3.7), one immediately gets,

$$\left(\Box_{NC} - M^2\right)\delta A^{+\{m_1,\ldots,m_p\}}_\alpha(y) = 0\,, \tag{3.13}$$

with,

$$M^2 = 2i\sum_{\beta=1}^p (2m_\beta+1)f_\beta - 4if_\alpha\,, \tag{3.14}$$

and

$$\left(\Box_{NC} - M^2\right)\delta A^{+\{m_1,\ldots,m_p\}}_{\bar\alpha}(y) = 0\,, \tag{3.15}$$

with,

$$M^2 = 2i\sum_{\beta=1}^p (2m_\beta+1)f_\beta + 4if_\alpha\,, \tag{3.16}$$

which indeed agrees with the leading term in eq. (3.1). In the remainder of the paper, we will concentrate on the spectrum of δA^+_α and denote it simply by δA_α. It is a trivial exercise to extend the results to $\delta A^+_{\bar\alpha}$.

3.2 Lower order results

For $\mathcal{L}_0 + \mathcal{L}_2$, the linearized equations of motion become,

$$0 = \left[\left(1 + \frac{1}{3} f_\alpha^2 - \frac{1}{6} \sum_\gamma f_\beta^2 \right) \mathcal{D}^2 + \frac{2}{3} \sum_\beta f_\beta^2 \left(\mathcal{D}_\beta \mathcal{D}_{\bar\beta} - i f_\beta - i f_\alpha \right) + 4i \left(f_\alpha + \frac{2}{3} f_\alpha^3 \right) \right] \delta A_\alpha -$$
$$- \sum_\beta \left[1 + \frac{1}{3}(f_\alpha^2 + f_\beta^2) - \frac{1}{6} \sum_\gamma f_\gamma^2 \right] \mathcal{D}_\alpha (\mathcal{D}_\beta \delta A_{\bar\beta} + \mathcal{D}_{\bar\beta} \delta A_\beta) \,, \tag{3.17}$$

with $\mathcal{D}^2$ given in eq. (3.5). As the linearized equation of motion should be of the form $(\Box_{NC} + \cdots)\delta A_\alpha = 0$, we need to make a field redefinition,

$$\delta \hat{A}_\alpha = \left(1 + \frac{1}{3} f_\alpha^2 - \frac{1}{6} \sum_\beta f_\beta^2 \right) \delta A_\alpha \,. \tag{3.18}$$

Using this and eq. (3.7), we can rewrite eq. (3.17) as

$$0 = \left[\Box_{NC} + 2 \sum_\beta \left(1 + \frac{1}{3} f_\beta^2 \right) \left(\mathcal{D}_\beta \mathcal{D}_{\bar\beta} - i f_\beta \right) + 4i \left(f_\alpha + \frac{1}{3} f_\alpha^3 \right) \right] \delta \hat{A}_\alpha -$$
$$- \frac{2}{9} \left[\left(f_\alpha^2 - \frac{1}{2} \sum_\gamma f_\gamma^2 \right) \sum_\beta f_\beta^2 \left(\mathcal{D}_\beta \mathcal{D}_{\bar\beta} - i f_\beta \right) + 2i f_\alpha^3 \left(f_\alpha^2 - \frac{1}{2} \sum_\beta f_\beta^2 \right) \right] \delta A_\alpha -$$
$$- \sum_\beta \left[1 + \frac{1}{3} \left(f_\alpha^2 + f_\beta^2 \right) - \frac{1}{6} \sum_\gamma f_\gamma^2 \right] \mathcal{D}_\alpha \left(\mathcal{D}_\beta \delta A_{\bar\beta} + \mathcal{D}_{\bar\beta} \delta A_\beta \right) . \tag{3.19}$$

The first line is precisely what we need. Indeed, proceeding as in the previous section, one finds that the spectrum of $\delta \hat{A}_\alpha$ reproduces eq. (3.1) through order $F^3 = (i\, f)^3$. The second line in eq. (3.19) can presently be ignored as it will contribute order f^5 corrections to the spectrum. However, these terms will interfere with the contributions arising from $\mathcal{L}_4$ (see the analysis in the next section). Finally, the last line of eq. (3.19) can be eliminated by making an appropriate gauge choice,

$$\sum_\beta \left(1 + \frac{1}{3} f_\beta^2 \right) \left(\mathcal{D}_\beta \delta A_{\bar\beta} + \mathcal{D}_{\bar\beta} \delta A_\beta \right) = 0 \,. \tag{3.20}$$

Note that this again yields terms which should be taken into account when analyzing the $\mathcal{L}_4$ contributions.

The α'^3 term, eq. (1.4), results in the following linearized equations of motion,

$$0 = -\frac{4\zeta(3)}{\pi^3} \left[\left(f_\alpha^2 \mathcal{D}^2 - 2 \sum_\beta f_\beta^2 (\mathcal{D}_\beta \mathcal{D}_{\bar\beta} - i f_\beta) \right) \left(\mathcal{D}^2 + 4i f_\alpha \right) \right] \delta A_\alpha +$$
$$+ \frac{4\zeta(3)}{\pi^3} \sum_\beta \mathcal{D}_\alpha \left[(f_\alpha^2 + f_\beta^2) \mathcal{D}^2 - 4 \sum_\gamma f_\gamma^2 (\mathcal{D}_\gamma \mathcal{D}_{\bar\gamma} - i f_\gamma) \right] (\mathcal{D}_\beta \delta A_{\bar\beta} + \mathcal{D}_{\bar\beta} \delta A_\beta) \,. \tag{3.21}$$

Using eq. (3.21) one can see that the first line of eq. (3.19) still holds for $\mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_3$, if we now take $\delta\hat{A}_\alpha$ to be,

$$\delta\hat{A}_\alpha = \left[1 + \frac{1}{3}f_\alpha^2\left(1 - \frac{12\zeta(3)}{\pi^3}\mathcal{D}^2\right) - \frac{1}{6}\sum_\beta f_\beta^2\left(1 - \frac{48\zeta(3)}{\pi^3}(\mathcal{D}_\beta\mathcal{D}_{\bar\beta} - if_\beta)\right)\right]\delta A_\alpha\,, \quad (3.22)$$

while modifying the gauge condition (3.20) to,

$$\sum_\beta\left[1 + \frac{1}{3}f_\beta^2\left(1 - \frac{12\zeta(3)}{\pi^3}\mathcal{D}^2\right)\right]\left(\mathcal{D}_\beta\delta A_{\bar\beta} + \mathcal{D}_{\bar\beta}\delta A_\beta\right) = 0\,. \quad (3.23)$$

This will introduce additional terms in the spectrum of order f^6 which will interfere with contributions coming from $\mathcal{L}_5$. As the analysis of the present paper is limited to $\mathcal{L}_4$ ($\mathcal{L}_5$ is not even known), we can safely ignore them.

Concluding, we find that $\mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_3$ correctly reproduces the spectrum, eq. (3.1), through this order.

3.3 The order α'^4 result

We now turn to the main point of the present paper: the contributions to the spectrum which arise from $\mathcal{L}_4$.

As the order increases, the calculations become rather tedious, one of the reasons being the symmetrized trace prescription. It turns out, however, that in our particular case one can very easily reduce the symmetrized trace to an ordinary trace.

Since we are only interested in the linearized form of the equations of motion, we only need to consider the symmetrized trace of a product of matrices, of which at most two have off-diagonal components.[4] A convenient way to do this was proposed in [19]. For our purpose, their more general formula simplifies in the following way: consider a product of $2n$ abelian fieldstrengths $\mathcal{F}_m$, $m \in \{1, \ldots, 2n\}$ given by,

$$\mathcal{F}_m = i\begin{pmatrix} F_m & 0 \\ 0 & -F_m \end{pmatrix}, \quad (3.24)$$

and two arbitrary two by two matrices with only off-diagonal components, which we call G and H,

$$G = i\begin{pmatrix} 0 & G^+ \\ G^- & 0 \end{pmatrix}, \qquad H = i\begin{pmatrix} 0 & H^+ \\ H^- & 0 \end{pmatrix}. \quad (3.25)$$

Then we have,

$$\mathrm{STr}\left(GH\mathcal{F}_1\cdots\mathcal{F}_{2n}\right) = \frac{(-1)^n}{2n+1}F_1\cdots F_{2n}\mathrm{Tr}(GH)\,. \quad (3.26)$$

We see that in this case, taking a symmetrized trace is no more difficult than taking an ordinary trace.

[4] In case only one of them is off-diagonal, the operations STr and Tr coincide, so we only consider the other case.

Using this result, the linearized equations of motion coming from $\mathcal{L}_{4,0}$ are still easy to obtain and are given by,

$$
\begin{aligned}
0 = -\frac{1}{5}\Bigg[&\sum_\beta\left(\frac{1}{4}f_\beta^4+\frac{1}{2}f_\alpha^2 f_\beta^2-\frac{1}{8}f_\beta^2\sum_\gamma f_\gamma^2\right)\mathcal{D}^2+if_\alpha\sum_\beta\left(4f_\alpha^2 f_\beta^2+f_\beta^4-\frac{1}{2}f_\beta^2\sum_\gamma f_\gamma^2\right)-\\
&-2\sum_\beta\left(f_\beta^4+f_\alpha^2 f_\beta^2-\frac{1}{2}f_\beta^2\sum_\gamma f_\gamma^2\right)(\mathcal{D}_\beta\mathcal{D}_{\bar\beta}-if_\beta)-f_\alpha^4\mathcal{D}^2-12if_\alpha^5\Bigg]\delta A_\alpha-\\
&-\frac{1}{5}\sum_\beta\left[f_\alpha^4+f_\beta^4+f_\alpha^2 f_\beta^2-\frac{1}{2}\sum_\gamma\left(\frac{1}{2}f_\gamma^4+f_\alpha^2 f_\gamma^2+f_\beta^2 f_\gamma^2-\frac{1}{4}f_\gamma^2\sum_\delta f_\delta^2\right)\right]\times\\
&\times\mathcal{D}_\alpha\left(\mathcal{D}_\beta\delta A_{\bar\beta}+\mathcal{D}_{\bar\beta}\delta A_\beta\right).
\end{aligned}
\tag{3.27}
$$

This expression would lead to a correction to the mass spectrum which is easily shown to deviate from eq. (3.1). This explicitely demonstrates, as was known long before [18, 19] that from this order on, the symmetrized trace prescription should receive corrections. After quite a lengthy calculation we find the linearized equations of motion for $\mathcal{L}_{4,2}+\mathcal{L}_{4,4}$ to be,

$$
\begin{aligned}
0 = \Bigg\{\Bigg[&\frac{1}{180}if_\alpha^3\mathcal{D}^2+\frac{4}{15}f_\alpha^4+\sum_\beta\Bigg(\frac{1}{45}if_\alpha f_\beta^2\mathcal{D}^2-\frac{7}{45}if_\alpha f_\beta^2(\mathcal{D}_\beta\mathcal{D}_{\bar\beta}-if_\beta)-\frac{7}{90}f_\beta^4-\frac{1}{5}f_\alpha^2 f_\beta^2+\\
&+\frac{1}{18}f_\beta^2\sum_\gamma f_\gamma^2\Bigg)\Bigg]\mathcal{D}^2+\sum_\beta\left(\frac{4}{9}f_\alpha^2 f_\beta^2+\frac{4}{45}f_\beta^2\sum_\gamma f_\gamma^2\right)\times\\
&\times(\mathcal{D}_\beta\mathcal{D}_{\bar\beta}-if_\beta)+\frac{4}{5}if_\alpha^5-if_\alpha\sum_\beta\left(\frac{14}{45}f_\beta^4-\frac{8}{15}f_\alpha^2 f_\beta^2-\frac{2}{9}f_\beta^2\sum_\gamma f_\gamma^2\right)\Bigg\}\delta A_\alpha-\\
&-\mathcal{D}_\alpha\sum_\beta\Bigg[\left(\frac{1}{180}if_\alpha^3-\frac{1}{18}if_\alpha f_\beta^2+\frac{1}{36}if_\alpha\sum_\gamma f_\gamma^2\right)\mathcal{D}^2-\frac{2}{45}if_\alpha\sum_\gamma f_\gamma^2(\mathcal{D}_\gamma\mathcal{D}_{\bar\gamma}-if_\gamma)-\\
&-\frac{4}{45}f_\alpha^2 f_\beta^2-\frac{4}{45}\sum_\gamma\left(f_\gamma^4-f_\beta^2 f_\gamma^2+f_\alpha^2 f_\gamma^2\right)\Bigg](\mathcal{D}_\beta\delta A_{\bar\beta}+\mathcal{D}_{\bar\beta}\delta A_\beta)+\\
&+\mathcal{D}_\alpha\sum_\beta\left(\frac{1}{360}f_\alpha f_\beta\mathcal{D}^2-\frac{1}{18}if_\alpha^2 f_\beta+\frac{1}{180}if_\beta^3+\frac{1}{36}if_\beta\sum_\gamma f_\gamma^2\right)\mathcal{D}^2\left(\mathcal{D}_\beta\delta A_{\bar\beta}-\mathcal{D}_{\bar\beta}\delta A_\beta\right).
\end{aligned}
\tag{3.28}
$$

The sum of the righthand sides of eqs. (3.27) and (3.28), together with the contributions coming from the second line of eq. (3.19) and those arising from the gauge choice, eq. (3.23), should now produce the correct F^5 terms in eq. (3.1). Careful analysis shows that the linearized equations of motion corresponding to the total lagrangian, eqs. (1.2)–(1.6), can eventually be written as,

$$
\left[\Box_{NC}+2\sum_\beta(1+\frac{1}{3}f_\beta^2+\frac{1}{5}f_\beta^4)(\mathcal{D}_\beta\mathcal{D}_{\bar\beta}-if_\beta)+4i(f_\alpha+\frac{1}{3}f_\alpha^3+\frac{1}{5}f_\alpha^5)\right]\delta\hat{A}_\alpha=0\,,
\tag{3.29}
$$

where, again, terms at fifth and higher order are ignored. The total correction to the eigenvectors δA_α, which appear in eq. (3.29), is given by,

$$\begin{aligned}\delta\hat{A}_\alpha = &\left[1+\frac{1}{3}f_\alpha^2\left(1-\left(\frac{12\zeta(3)}{\pi^3}-\frac{1}{60}if_\alpha\right)\mathcal{D}^2+\frac{22}{15}f_\alpha^2-\frac{1}{30}\sum_\gamma f_\gamma^2\right)-\right.\\ &-\frac{1}{6}\sum_\beta f_\beta^2\left(1-\left(\frac{48\zeta(3)}{\pi^3}-\frac{14}{15}if_\alpha\right)\left(\mathcal{D}_\beta\mathcal{D}_{\bar\beta}-if_\beta\right)-\frac{2}{15}if_\alpha\mathcal{D}^2+\right.\\ &\left.\left.+\frac{23}{30}f_\beta^2-\frac{29}{60}\sum_\gamma f_\gamma^2\right)\right]\delta A_\alpha+ \qquad (3.30)\\ &+\sum_\beta\mathcal{D}_\alpha\left(\frac{1}{360}f_\alpha f_\beta\mathcal{D}^2-\frac{1}{18}if_\alpha^2 f_\beta+\frac{1}{180}if_\beta^3+\frac{1}{36}if_\beta\sum_\gamma f_\gamma^2\right)\left(\mathcal{D}_\beta\,\delta A_{\bar\beta}-\mathcal{D}_{\bar\beta}\,\delta A_\beta\right).\end{aligned}$$

The gauge condition (3.23) also gets fourth order contributions and becomes,

$$\sum_\beta\left[1+\frac{1}{3}f_\beta^2\left(1-\frac{12\zeta(3)}{\pi^3}\mathcal{D}^2+\frac{2}{15}\sum_\gamma f_\gamma^2\right)+\frac{1}{5}f_\beta^4\right]\left(\mathcal{D}_\beta\delta A_{\bar\beta}+\mathcal{D}_{\bar\beta}\delta A_\beta\right)=0\,. \qquad (3.31)$$

Eq. (3.29) exactly leads to the mass spectrum in eq. (3.1). This shows that, if we redefine the mass eigenvectors δA_α in an appropriate way and impose the right gauge condition, we obtain total agreement with string theoretical calculations up to fourth order in α'!

4. Discussion

The bosonic terms in the non-abelian D-brane effective action are known through order α'^4, [16] (the fermionic terms are only known through order α'^3 [11]). In the present paper, we performed a successful test of this result. Indeed when switching on constant magnetic background fields, we showed that through this order, the spectrum agrees with the one obtained from a direct string theoretical calculation. The contributions coming from the symmetrized trace part of the lagrangian, $\mathcal{L}_{4,0}$, combined with those arising from the derivative terms in the action, $\mathcal{L}_{4,2}+\mathcal{L}_{4,4}$, and those which arose from $\mathcal{L}_2$ as a consequence of the field redefinition and the gauge choice, precisely reproduce the α'^4 terms in the spectrum, eq. (3.1). However, we would like to stress that this does not check every coefficient in the action. Indeed, when going through the details of the calculation, one finds e.g. that the last term in $\mathcal{L}_{4,2}$ and the second and the last term in $\mathcal{L}_{4,4}$ do not contribute at all.

Nonetheless, combining this test with the fact that the calculation of α'^4 term in [16] required solving 1816 algebraic equations in 546 unknowns yielding a unique solution, shows that we can be very confident about the results in [16].

Yet another test is provided by the results in [28] (see also [29]). Requiring supersymmetry, the derivative terms in the abelian theory were determined through order α'^4. While this method does not fix the overall constant in front of these terms (they form an independent supersymmetry invariant), the relative coefficients are fixed. Taking the abelian limit of eq. (1.6) gives a result which perfectly agrees with the one in [28], however in our case the overall coefficient is fixed.

Eqs. (1.4) and (1.6) are very involved. At first sight there seems to be little hope that a closed expression to all orders in α' can be found. The possibility of making field redefinitions further complicates matters. We are convinced that as a first step, the derivative corrections in the abelian limit should be investigated. If there is any organizational principle for the non-abelian effective action to all orders in α', this should be true for the full abelian effective action, which obviously is much simpler, as well. This is presently being studied.

Finally, in [30], the recombination of intersecting D1-branes was analyzed using the leading term in the non-abelian D-brane effective action by studying the tachyonic configurations. While the analysis of [30] is performed in a gauge different from ours, it is straightforward using eqs. (3.19) and (3.18) to repeat their analysis through second order. No essential new features are added to their conclusions. However, from third order on, the field redefinitions are more subtle as they involve derivative terms as well. So it would be interesting, after T-dualizing the results given in previous sections, to study higher order effects on D-string recombination along the lines of [30] including the corrections through order α'^4.

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